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GENERAL THEORY OF SANDWICH PLATES WITH DISSIMILAR FACINGS

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Abstract—Equations are derived for the large deflections of sandwich plates with weak cores and presented in an invariant form. The general equations include the bending resistance of the facings and transverse extension of the core. Equations for buckling and for small deflections are obtained as special cases. An example illustrates the use of these equations to predict buckling loads.

NOTATION

The usual suffix notations are used. Latin suffixes represent the numbers 1, 2 and 3, while Greek suffixes represent only 1 and 2. Repeated suffixes imply summation unless enclosed by parentheses.

Since there is no need to denote covariant differentiation with respect to the three dimensional space, the vertical line (|) is used here to denote covariant differentiation with respect to the undeformed middle surface. A comma (,) denotes partial differentiation.

A prefix $\underline{n} = \underline{0}$ or $\underline{1}$ refers to the upper or lower facing; the prefix is underlined to avoid confusion with a suffix. Upper and lower signs, e.g. $\pm$, apply accordingly as the prefix $\underline{n} = \underline{0}$ or $\underline{1}$, respectively.

Where a distinction is necessary, capital letters refer to the deformed plate while lower-case letters refer to the undeformed plate.

Essentially new variables are defined where they arise. Others are listed below.

L	a characteristic length of the middle surface
d	thickness of the core
${}_{\underline{n}}d$	thickness of a facing ($\underline{n} = \underline{0}$ or $\underline{1}$)
λ	$= d/2L$
$\bar{\lambda}$	$= 2\lambda + {}_{\underline{0}}\lambda/2 + {}_{\underline{1}}\lambda/2$
${}_{\underline{n}}\lambda$	$= {}_{\underline{n}}d/L$
$\tilde{\lambda}$	$= {}_{\underline{0}}\lambda - {}_{\underline{1}}\lambda$
θ^α	dimensionless surface coordinate
θ^3	$= \dfrac{x^3}{\lambda L}$; x^3 = length along the normal to the middle surface
$\vec{a}_\alpha$ (or $\vec{A}_\alpha$)	$= \vec{R}_{,\alpha}$; the position vector is $L\vec{R}$
$\vec{a}_3$ (or $\vec{A}_3$)	$= \vec{R}_{,3}$
$a_{\alpha\beta}$	$= \vec{a}_\alpha \cdot \vec{a}_\beta$
a	= determinant $\lvert a_{\alpha\beta} \rvert$
$\varepsilon_{\alpha\beta}$	$= \sqrt{(a)}e_{\alpha\beta}$; $e_{\alpha\beta}$ = permutation symbol
$\vec{c}$	$= \bar{c}_\alpha \vec{A}^\alpha$ = unit tangent to the edge

$\vec{u}$	$= \bar{u}_\alpha \vec{A}^\alpha$ = unit normal to the edge in the surface
g_{ij}	= metric tensor associated with θ^i
g	= determinant $\lvert g_{ij}\rvert$
${}_n\vec{p}$	$= \frac{1}{L^2}({}_np_n^\beta \vec{A}_\beta + {}_np_n^3 \vec{A}_3)$, external force per unit area
${}_n\vec{N}^\alpha$	$= \sqrt{(a)}({}_nn_n^{\alpha\gamma}\vec{A}_\gamma + {}_nq_n^\alpha \vec{A}_3)$, force resultant per unit of the θ^β coordinate line ($\beta \neq \alpha$)
${}_n\vec{M}^\alpha$	$= \sqrt{(a)}L\varepsilon_{\gamma\rho n}m_n^{\alpha\gamma}\vec{A}^\rho$, resultant couple per unit of the θ^β coordinate line ($\beta \neq \alpha$)
${}_n\vec{V}$	$= {}_nv_i\vec{a}^i$, interface displacement
${}_0\vec{V} - {}_1\vec{V}$	$= 2w_i\vec{a}^i$, relative displacement of the interfaces
${}_0\vec{V} + {}_1\vec{V}$	$= 2\bar{w}_i\vec{a}^i$, average displacement of the interfaces
$\bar{p}^i$	$= {}_0p^i + {}_1p^i$
p^3	$= {}_0p^3 - {}_1p^3$
$\tilde{p}^\gamma$	$= {}_0p^\gamma - \frac{1}{\gamma}{}_1p^\gamma$
$\bar{c}^\alpha$	$= {}_0\lambda_0 p^\alpha - {}_1\lambda_1 p^\alpha$
c^α	$= \lambda({}_0\lambda_0 p^\alpha + {}_1\lambda_1 p^\alpha)$
$\bar{n}^{\alpha\beta}$	$= {}_0n^{\alpha\beta} + {}_1n^{\alpha\beta}$
$\tilde{n}^{\alpha\beta}$	$= \lambda\left({}_0n^{\alpha\beta} - \frac{1}{\gamma}{}_1n^{\alpha\beta}\right)$
$\bar{m}^{\alpha\beta}$	$= {}_0m^{\alpha\beta} + {}_1m^{\alpha\beta}$
$m^{\alpha\beta}$	$= \lambda({}_0m^{\alpha\beta} - {}_1m^{\alpha\beta})$
$\bar{q}^\alpha$	$= {}_0q^\alpha + {}_1q^\alpha = \bar{m}^{\gamma\alpha}\rvert_\gamma + \bar{c}^\alpha$
q^α	$= \lambda({}_0q^\alpha - {}_1q^\alpha) = m^{\gamma\alpha}\rvert_\gamma + c^\alpha$
${}_nB_{\alpha\beta}$	$= \frac{1}{L}(\bar{w}_3 \pm w_3)\rvert_{\alpha\beta}$
$\bar{B}_{\alpha\beta}$	$= \frac{1}{2}({}_0B_{\alpha\beta} + {}_1B_{\alpha\beta})$
$B_{\alpha\beta}$	$= \frac{1}{2}({}_0B_{\alpha\beta} - {}_1B_{\alpha\beta})$
${}_n\check{\gamma}_{\alpha\beta}$	strain of an interface
${}_n\gamma_{\alpha\beta}$	strain of the middle surface of a facing
$\bar{\gamma}_{\alpha\beta}$	$= ({}_0\gamma_{\alpha\beta} + \gamma_1\gamma_{\alpha\beta})$
τ^{ij}	stress tensor
$\bar{s}^\alpha$	$= \frac{\lambda L}{\sqrt{a}}\int_{-1}^{+1}\sqrt{(g)}\tau^{3\alpha}\,d\theta^3$
σ^{33}	$= \frac{L^2\lambda^3}{\sqrt{a}}(\sqrt{({}_0a)}{}_0\tau^{33} + \sqrt{({}_1a)}{}_1\tau^{33})$
ϕ	an invariant stress function; cf. equation (16)
$\bar{F}^{\alpha\beta}$	see equation (17)
$\bar{P}$	$= LB\left(\bar{p}^3 + \frac{1}{2}\bar{c}^\alpha\rvert_\alpha + \frac{\bar{\lambda}\gamma}{1+\gamma}\tilde{p}^\alpha\rvert_\alpha\right) - \frac{(1-\eta)}{2L\bar{\lambda}_0\lambda_0\mu}\tilde{p}^\alpha\Big\rvert^\beta_{\alpha\beta}$
χ	an invariant 'stress function'
$\tilde{P}$	see equation (24c)
$\bar{\bar{P}}$	$= L\left[\bar{p}^3 + \frac{1}{2}\bar{c}^\beta\rvert_\beta - \frac{4\lambda_0\lambda_0\mu\gamma}{(1-\eta)G(1+\gamma)}\left(\bar{p}^3\Big\rvert^\alpha_\alpha + \frac{1}{2}\bar{c}^\beta\Big\rvert^\alpha_{\beta\alpha}\right) + \frac{\bar{\lambda}\gamma}{1+\gamma}\tilde{p}^\beta\Big\rvert_\beta\right]$
A	$= \frac{3(1-\eta)G\bar{\lambda}^2}{({}_0\lambda^2 + \gamma_1\lambda^2)\lambda_0\lambda_0\mu} + \frac{(1-\eta)G}{4\lambda_0\lambda_0\mu}\,\frac{1+\gamma}{\gamma}$
B	$= \frac{3(1-\eta)^2G(1+\gamma)}{2L^2({}_0\lambda^2 + \gamma_1\lambda^2)\lambda_0\lambda^2{}_0\mu^2\gamma}$
C	$= \frac{4\lambda_0\lambda_0\mu}{(1-\eta)G}\left(\frac{\gamma}{1+\gamma}\right)$
$\bar{X}_i, X_i$	physical (not tensorial) components of edge forces
$\bar{H}_\alpha, H_\alpha$	physical (not tensorial) components of edge couples
$\mathscr{S}$	edge shear force on the core
$\bar{\Delta}_i, \Delta_i$	physical (not tensorial) components of edge displacements
η	Poisson's ratio of both facings

${}_n\mu$	shear modulus of a facing
${}_nE$	Young's modulus of a facing
G^α	transverse shear modulus of an orthotropic core
G	transverse shear modulus of an isotropic core
E	Young's modulus for transverse extension of core
$\bar{B}^{\alpha\beta\gamma\eta}$	stiffness tensor; cf. equation (20b)
$\bar{C}_{\alpha\beta\gamma\eta}$	flexibility tensor; cf. equation (20a)
$C_{\alpha\beta}$	shear flexibility tensor of core; cf. equation (2c)
γ	$= \dfrac{{}_1\lambda_1\mu}{{}_0\lambda_0\mu}$

INTRODUCTION

MUCH attention has been directed to the behavior of sandwich plates. Notable contributions to the theory were made by Reissner [1, 2, 3], Libove and Batdorf [4], Wang [5] and Yu [6]. Numerous others have dealt with the stability of sandwich plates, but much of the work does not account for the bending resistance of the individual facing. Equations which include the bending stiffness of the facings were given by Grigolyuk [7] and elaborated by Fulton [8]. They do not include the transverse extensibility of the core. Equations for rectangular plates with equal facings were given by Prusakov [9].

In a previous paper [10] the authors presented a general theory of sandwich shells with weak cores. The basic theory of that paper is applicable to sandwich plates. However, the formulation of the earlier work is not well suited to the analysis of plates with dissimilar facings. Moreover, the specialization from a curved surface to a plane simplifies the equations and enables us to introduce invariant functions which reduce the number of dependent variables and differential equations. Here we present a general formulation which includes dissimilarities in the facings and the important geometrical non-linearities. The equations are developed from a unified point of view with few approximations; the important approximations are (1) the neglect of membrane and bending stresses in the core and (2) the Kirchhoff hypothesis in the facings. The geometrical nonlinearities are comparable to those of the von Kármán theory of one-layer plates. Orthotropy of the core and facings is taken into account. The nonlinear formulation is expressed by five partial differential equations or two if the plate is isotropic and also transversely inextensible. Equations for small deflections and equations for buckling are obtained as special cases.

The equations are applied to the buckling of plates with unequal facings.

SOME BASIC EQUATIONS

We begin by citing the basic results from the earlier paper [10]. Notations have previously been described.

The stress distributions in the core are

$$\tau^{3\alpha} = \frac{\bar{s}^\alpha}{2\lambda^2 L^4}, \tag{1a}$$

$$\tau^{33} = -\frac{\bar{s}^\alpha|_\alpha}{2\lambda^2 L^4}\theta^3 + \frac{\sigma^{33}}{2\lambda^3 L^2}, \tag{1b}$$

wherein $\bar{s}^\alpha$ and σ^{33} are a transverse shear resultant and mean normal stress.

The core behavior is described by relating the relative interface displacements w_i to the dynamic variables $\bar{s}^\alpha$ and σ^{33}. The relations are

$$w_3 = \frac{L}{2E}\sigma^{33} - \frac{a^{\alpha\beta}}{2\lambda L^3}\bar{\bar{\omega}}_{3\alpha}\bar{\bar{\omega}}_{3\beta}, \tag{2a}$$

$$w_\alpha = -\lambda \bar{w}_3|_\alpha + \frac{C_{\alpha\beta}}{2L}\bar{s}^\beta - \frac{\lambda^2}{6LE}\bar{s}^\beta|_{\beta\alpha}. \tag{2b}$$

If the core is orthotropic

$$C_{\alpha\beta} = \frac{a_{\alpha\beta}}{G^{(\alpha)}} \tag{2c}$$

G^α and E are the transverse shear and extension moduli, $\bar{w}_i$ is the mean of the interface displacements and $\bar{\bar{\omega}}_{3\alpha}$ is a gross rotation,

$$\bar{\bar{\omega}}_{3\alpha} = \frac{\lambda L}{2}\bar{w}_3|_\alpha - \frac{L}{2}w_\alpha.$$

In the quadratic term of (2a) the transverse shear strain is assumed small compared to the rotation $\bar{\bar{\omega}}_{3\alpha}$; then it is consistent to use the approximation $w_\alpha = -\lambda\bar{w}_3|_\alpha$ to obtain

$$\bar{\bar{\omega}}_{3\alpha} = \lambda L \bar{w}_3|_\alpha. \tag{2d}$$

It is worth noting that (1a, b) and (2a, b) are exact within the limitations of linear theory and the weak-core hypothesis ($\tau^{\alpha\beta} = 0$).

The second fundamental tensor of the deformed interface is

$$_{\underline{n}}B_{\alpha\beta} = \frac{1}{L}(\bar{w}_3 \pm w_3)|_{\alpha\beta}. \tag{3a}$$

The strain components at the interfaces are given by

$$_{\underline{n}}\check{\gamma}_{\alpha\beta} = \frac{L}{2}(\bar{w}_\alpha|_\beta + \bar{w}_\beta|_\alpha \pm w_\alpha|_\beta \pm w_\beta|_\alpha) + \tfrac{1}{2}(\bar{w}_3|_\alpha \pm w_3|_\alpha)(\bar{w}_3|_\beta \pm w_3|_\beta). \tag{3b}$$

Here as elsewhere the products of rotations about a normal are neglected (see [10]). In accordance with the Kirchhoff hypothesis the strain components at the middle surface of a facing are given by

$$_{\underline{n}}\gamma_{\alpha\beta} = {}_{\underline{n}}\check{\gamma}_{\alpha\beta} \mp \frac{L^2{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}B_{\alpha\beta}. \tag{3c}$$

If the facings are linearly elastic and similar

$$\tau^{\alpha\beta} = \frac{_{\underline{n}}\mu}{L^4}\bar{B}^{\alpha\beta\rho\lambda}\gamma_{\rho\lambda},$$

where $_{\underline{n}}\mu$ is an elastic constant for the facing. Then the tensions and bending couples are related to the strain and curvature components by

$$_{\underline{n}}n^{\alpha\beta} = {}_{\underline{n}}\lambda\,{}_{\underline{n}}\mu\bar{B}^{\alpha\beta\gamma\eta}{}_{\underline{n}}\gamma_{\gamma\eta}, \tag{4a}$$

$$_{\underline{n}}m^{\alpha\beta} = -\frac{_{\underline{n}}\lambda^3L^2}{12}{}_{\underline{n}}\mu\bar{B}^{\alpha\beta\rho\lambda}{}_{\underline{n}}B_{\rho\lambda}. \tag{4b}$$

These forces and couples act at the middle surface of the facing.

When the shear resultants have been eliminated, the equations of motion for a facing are

$$ {}_{\underline{n}}p^3 \pm \frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}p^\alpha|_\alpha + \frac{1}{2}\left(1+\frac{{}_{\underline{n}}\lambda}{2\lambda}\right)\bar{s}^\beta|_\beta \mp \frac{L^2}{2\lambda}\sigma^{33} + {}_{\underline{n}}m^{\alpha\beta}|_{\alpha\beta} + {}_{\underline{n}}n^{\alpha\beta}{}_{\underline{n}}B_{\alpha\beta} = 0, \tag{5a} $$

$$ \left(\delta^\lambda_\alpha \mp \underset{\sim\sim}{\frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}B^\lambda_\alpha}\right){}_{\underline{n}}p^\alpha \mp \frac{1}{2\lambda}\left(\delta^\lambda_\alpha \pm \underset{\sim\sim}{\frac{{}_{\underline{n}}\lambda}{2}{}_{\underline{n}}B^\lambda_\alpha}\right)\bar{s}^\alpha + {}_{\underline{n}}n^{\alpha\lambda}|_\alpha - \underset{\sim\sim\sim}{{}_{\underline{n}}B^\lambda_{\alpha\underline{n}}m^{\alpha\beta}|_\beta} = 0. \tag{5b} $$

These equations are exact and applicable to dynamic situations when the load components ${}_{\underline{n}}p^i$ include the inertial loads. The underlined terms of (5b) can be neglected; then it reduces to

$$ {}_{\underline{n}}p^\beta \mp \frac{1}{2\lambda}\bar{s}^\beta + {}_{\underline{n}}n^{\alpha\beta}|_\alpha = 0. \tag{6a} $$

A COMPATIBILITY CONDITION

The Gauss equation of the undeformed and deformed middle surfaces of a facing are (see [11])

$$ \tfrac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\eta}{}_{\underline{n}}r_{\alpha\beta\gamma\eta} = 0 \tag{7a} $$

and

$$ \tfrac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\eta}{}_{\underline{n}}R_{\alpha\beta\gamma\eta} = \frac{A}{a}{}_{\underline{n}}K, \tag{7b} $$

where ${}_{\underline{n}}r_{\alpha\beta\gamma\eta}$ and ${}_{\underline{n}}R_{\alpha\beta\gamma\eta}$ are the Riemann–Christoffel tensors of the respective surfaces and ${}_{\underline{n}}K$ is the Gaussian curvature of the deformed surface. If (7a) is subtracted from (7b) and products of the strain components are neglected, the result is

$$ \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}{}_{\underline{n}}\gamma_{\alpha\delta}|_{\beta\gamma} = L^2{}_{\underline{n}}K = \frac{L^2}{2}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}{}_{\underline{n}}B_{\alpha\gamma\underline{n}}B_{\beta\delta}. \tag{8} $$

We define a weighted strain component,

$$ \bar{\gamma}_{\alpha\beta} = {}_{\underline{0}}\gamma_{\alpha\beta} + \gamma\,{}_{\underline{1}}\gamma_{\alpha\beta} \tag{9} $$

wherein

$$ \gamma = \frac{{}_{\underline{1}}\lambda_{\underline{1}}\mu}{{}_{\underline{0}}\lambda_{\underline{0}}\mu}. \tag{10} $$

To obtain the desired compatibility equation we multiply ${}_{\underline{0}}(8)$ by ${}_{\underline{0}}\lambda_{\underline{0}}\mu$, ${}_{\underline{1}}(8)$ by ${}_{\underline{1}}\lambda_{\underline{1}}\mu$, then add them, use (3a), and obtain

$$ \begin{aligned} {}_{\underline{0}}\lambda_{\underline{0}}\mu\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\bar{\gamma}_{\alpha\delta}|_{\beta\gamma} = \tfrac{1}{2}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}[({}_{\underline{0}}\lambda_{\underline{0}}\mu + {}_{\underline{1}}\lambda_{\underline{1}}\mu)\,.\,(\bar{w}_3|_{\alpha\gamma}\bar{w}_3|_{\beta\delta} + w_3|_{\alpha\gamma}w_3|_{\beta\delta}) + \\ + ({}_{\underline{0}}\lambda_{\underline{0}}\mu - {}_{\underline{1}}\lambda_{\underline{1}}\mu)(\bar{w}_3|_{\alpha\gamma}w_3|_{\beta\delta} + w_3|_{\alpha\gamma}\bar{w}_3|_{\beta\delta})] = \mathscr{F}(\bar{w}_3, w_3). \end{aligned} \tag{11} $$

EQUILIBRIUM CONDITIONS

Instead of the equilibrium equations (6) we form linear combinations as follows: The sum of ${}_{\underline{0}}(6a)$ and ${}_{\underline{1}}(6a)$ is

$$ \bar{p}^\beta + \bar{n}^{\alpha\beta}|_\alpha = 0, \tag{12a} $$

where bars ($^-$) denote a sum (see Notation). Next we divide ${}_0$(6a) by ${}_0\lambda_0\mu$, ${}_1$(6a) by ${}_1\lambda_1\mu$ and subtract the latter from the former. The result is

$$\tilde{n}^{\alpha\beta}|_\alpha - \frac{1+\gamma}{2\gamma}\bar{s}^\beta + \lambda\tilde{p}^\beta = 0 \tag{12b}$$

wherein

$$\tilde{n}^{\alpha\beta} = \lambda\left({}_0n^{\alpha\beta} - \frac{1}{\gamma}{}_1n^{\alpha\beta}\right),$$

$$\tilde{p}^\alpha = {}_0p^\alpha - \frac{1}{\gamma}{}_1p^\alpha.$$

By adding ${}_0$(5a) and ${}_1$(5a) we obtain

$$\bar{p}^3 + \frac{\bar{\lambda}}{2\lambda}\bar{s}^\beta|_\beta + \frac{1}{2}\bar{c}^\beta|_\beta + \bar{m}^{\alpha\beta}|_{\alpha\beta} + \bar{n}^{\alpha\beta}\bar{B}_{\alpha\beta} + \frac{1-\gamma}{1+\gamma}\bar{n}^{\alpha\beta}B_{\alpha\beta} + \frac{2\gamma}{\lambda(1+\gamma)}\tilde{n}^{\alpha\beta}B_{\alpha\beta} = 0. \tag{12c}$$

Herein

$$\bar{\lambda} = 2\lambda + \tfrac{1}{2}{}_0\lambda + \tfrac{1}{2}{}_1\lambda,$$

$$\bar{B}_{\alpha\beta} = \frac{1}{L}\bar{w}_3|_{\alpha\beta},\; B_{\alpha\beta} = \frac{1}{L}w_3|_{\alpha\beta}. \tag{13a, b}$$

By subtracting ${}_1$(5a) from ${}_0$(5a) we have

$$\lambda p^3 + \tfrac{1}{2}c^\beta|_\beta + \frac{\tilde{\lambda}}{4}\bar{s}^\beta|_\beta - L^2\sigma^{33} + m^{\alpha\beta}|_{\alpha\beta} + \frac{2\gamma}{1+\gamma}\tilde{n}^{\alpha\beta}\bar{B}_{\alpha\beta} + \lambda\frac{1-\gamma}{1+\gamma}\bar{n}^{\alpha\beta}\bar{B}_{\alpha\beta} + \lambda\bar{n}^{\alpha\beta}B_{\alpha\beta} = 0. \tag{12d}$$

BENDING AND STRETCHING RELATIONS

From (4a, b) we obtain

$$\bar{m}^{\alpha\beta} = -\frac{{}_0\lambda_0\mu L}{12}\bar{B}^{\alpha\beta\rho\lambda}[({}_0\lambda^2 + \gamma_1\lambda^2)\bar{w}_3|_{\rho\lambda} + ({}_0\lambda^2 - \gamma_1\lambda^2)w_3|_{\rho\lambda}], \tag{14a}$$

$$m^{\alpha\beta} = -\frac{{}_0\lambda_0\mu\lambda L}{12}\bar{B}^{\alpha\beta\rho\lambda}[({}_0\lambda^2 + \gamma_1\lambda^2)w_3|_{\rho\lambda} + ({}_0\lambda^2 - \gamma_1\lambda^2)\bar{w}_3|_{\rho\lambda}], \tag{14b}$$

$$\bar{n}^{\alpha\beta} = {}_0\lambda_0\mu\bar{B}^{\alpha\beta\rho\lambda}\bar{\gamma}_{\rho\lambda}, \tag{14c}$$

$$\tilde{n}^{\alpha\beta} = {}_0\lambda_0\mu\lambda L\bar{B}^{\alpha\beta\gamma\eta}\left(w_\gamma|_\eta + w_\eta|_\gamma - \frac{{}_0\lambda + {}_1\lambda}{2}\bar{w}_3|_{\gamma\eta} - \frac{\tilde{\lambda}}{2}w_3|_{\gamma\eta} + \frac{1}{L}w_3|_\gamma\bar{w}_3|_\eta + \frac{1}{L}\bar{w}_3|_\gamma w_3|_\eta\right). \tag{14d}$$

We can also replace the relative interface displacements w_α by the relative middle-surface displacements

$$\alpha_\gamma = 2w_\gamma - \frac{{}_0\lambda + {}_1\lambda}{2}\bar{w}_3|_\gamma - \frac{\tilde{\lambda}}{2}w_3|_\gamma. \tag{15}$$

Then (14d) is replaced by

$$\tilde{n}^{\alpha\beta} = \frac{{}_0\lambda_0\mu\lambda L}{2}\bar{B}^{\alpha\beta\gamma\eta}\left(\alpha_\gamma|_\eta + \alpha_\eta|_\gamma + \frac{2}{L}w_3|_\gamma\bar{w}_3|_\eta + \frac{2}{L}\bar{w}_3|_\gamma w_3|_\eta\right). \tag{14e}$$

THE STRESS FUNCTION

The equilibrium equation (12a) is identically satisfied if

$$\bar{n}^{\alpha\beta} = \varepsilon^{\alpha\eta}\varepsilon^{\beta\gamma}\phi|_{\gamma\eta} - \bar{F}^{\alpha\beta}, \tag{16}$$

where $\bar{F}^{\alpha\beta}$ is a symmetric particular integral of

$$\bar{F}^{\alpha\beta}|_\alpha = \bar{p}^\beta. \tag{17}$$

However $\bar{n}^{\alpha\beta}$ must derive from continuous displacements $\bar{w}_\alpha$ in accordance with (14c), (9) and (3). That is $\bar{\gamma}_{\alpha\beta}$ must satisfy the compatibility condition (11) and

$$\bar{\gamma}_{\alpha\beta} = \frac{1}{{}_0\lambda_0\mu}\bar{C}_{\alpha\beta\gamma\eta}\bar{n}^{\gamma\eta} \tag{18}$$

in which $\bar{C}_{\alpha\beta\gamma\eta}$ is the flexibility tensor, i.e.

$$\bar{C}_{\alpha\beta\rho\lambda}\bar{B}^{\rho\lambda\gamma\eta} = \bar{C}_{\alpha\beta\rho\lambda}\bar{B}^{\gamma\eta\rho\lambda} = \delta^\gamma_\alpha\delta^\eta_\beta.$$

In the sequel we will assume homogeneity so that $\bar{C}_{\alpha\beta\gamma\eta}|_\mu = 0$. Substituting (16) into (18) into (11) we obtain

$$\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\varepsilon^{\zeta\mu}\varepsilon^{\eta\xi}\bar{C}_{\alpha\delta\zeta\eta}\phi|_{\xi\mu\beta\gamma} - \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\bar{C}_{\alpha\delta\zeta\eta}\bar{F}^{\zeta\eta}|_{\beta\gamma} = \mathscr{F}(\bar{w}_3, w_3). \tag{19a}$$

If the facings are isotropic

$$\bar{C}_{\alpha\delta\zeta\eta} = \frac{1}{2}\left(a_{\alpha\zeta}a_{\delta\eta} - \frac{\eta}{1+\eta}a_{\zeta\eta}a_{\alpha\delta}\right). \tag{20a}$$

Then (19a) takes the form

$$\phi|^{\alpha\beta}_{\alpha\beta} + (1+\eta)\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\bar{F}_{\alpha\delta}|_{\beta\gamma} + \eta\bar{F}^\alpha_\alpha|^\beta_\beta = -2(1+\eta)\mathscr{F}(\bar{w}_3, w_3). \tag{19b}$$

THE NONLINEAR DIFFERENTIAL EQUATIONS

The core relation (2a) serves to eliminate the variable σ^{33} from (12d). The core relations (2b) serve to eliminate the relative displacements w_α (or α_α) from (14d); the result is

$$\tilde{n}^{\alpha\beta} = {}_0\lambda_0\mu\lambda\bar{B}^{\alpha\beta\gamma\eta}\Bigg(-\bar{\lambda}L\bar{w}_3|_{\gamma\eta} + C_{\gamma\mu}\bar{s}^\mu|_\eta - \frac{\lambda^2}{3E}\bar{s}^\mu|_{\mu\gamma\eta} - \frac{\tilde{\lambda}L}{2}w_3|_{\gamma\eta} + w_3|_\gamma\bar{w}_3|_\eta + \bar{w}_3|_\gamma w_3|_\eta\Bigg). \tag{21a}$$

When (21a), (14a, b), (16) and (13a, b) are substituted into (12b, c, d), the latter together with (19a) constitute a system of five partial differential equations in the five dependent variables $\bar{s}^\alpha$, ϕ, $\bar{w}_3$ and w_3.

If the plate is isotropic,

$$\bar{B}^{\alpha\beta\gamma\eta} = a^{\alpha\eta}a^{\beta\gamma} + a^{\alpha\gamma}a^{\beta\eta} + \frac{2\eta}{1-\eta}a^{\alpha\beta}a^{\gamma\eta} \tag{20b}$$

and

$$C_{\alpha\beta} = a_{\alpha\beta}/G.$$

Then

$$\begin{aligned} \tilde{n}^{\alpha\beta} = {}_0\lambda_0\mu\lambda\Bigg[& -2\bar{\lambda}L\left(\bar{w}_3|^{\alpha\beta} + \frac{\eta}{1-\eta}a^{\alpha\beta}\bar{w}_3|^{\eta}_{\eta}\right) \\ & + \frac{1}{G}\left(\bar{s}^{\beta}|^{\alpha} + \bar{s}^{\alpha}|^{\beta} + \frac{2\eta}{1-\eta}a^{\alpha\beta}\bar{s}^{\eta}|_{\eta}\right) \\ & - \frac{2\lambda^2}{3E}\left(\bar{s}^{\eta}|^{\alpha\beta}_{\eta} + \frac{\eta}{1-\eta}a^{\alpha\beta}\bar{s}^{\eta}|^{\mu}_{\eta\mu}\right) \\ & - \bar{\lambda}L\left(w_3|^{\alpha\beta} + \frac{\eta}{1-\eta}w_3|^{\eta}_{\eta}a^{\alpha\beta}\right) \\ & + 2\left(w_3|^{\alpha}\bar{w}_3|^{\beta} + \frac{\eta}{1-\eta}a^{\alpha\beta}w_3|^{\eta}\bar{w}_3|_{\eta}\right) \\ & + 2\left(\bar{w}_3|^{\alpha}w_3|^{\beta} + \frac{\eta}{1-\eta}a^{\alpha\beta}\bar{w}_3|^{\eta}w_3|_{\eta}\right)\Bigg]. \end{aligned} \tag{21b}$$

When (21b) is substituted into (12b), the latter is then covariantly differentiated and summed, the result is

$$\begin{aligned} & -2\bar{\lambda}L\bar{w}_3|^{\alpha\beta}_{\alpha\beta} + \frac{2}{G}\bar{s}^{\beta}|^{\alpha}_{\beta\alpha} - \frac{2\lambda^2}{3E}\bar{s}^{\alpha}|^{\beta\eta}_{\alpha\beta\eta} - \bar{\lambda}Lw_3|^{\alpha\beta}_{\alpha\beta} \\ & \quad - \frac{(1-\eta)(1+\gamma)}{2{}_0\lambda_0\mu\lambda\gamma}\bar{s}^{\beta}|_{\beta} + \frac{(1-\eta)}{{}_0\lambda_0\mu}\tilde{p}^{\beta}|_{\beta} \\ & \quad + 4(1-\eta)(w_3|^{\alpha}\bar{w}_3|^{\beta})|_{\alpha\beta} + 4\eta(\bar{w}_3|^{\eta}w_3|_{\eta})|^{\alpha}_{\alpha} = 0. \end{aligned} \tag{22a}$$

Notice that equation (22) contains $\bar{s}^{\alpha}$ only in the invariant $\bar{s}^{\alpha}|_{\alpha}$, but that $\bar{s}^{\alpha}$ will appear in a nonlinear term of (12c) and (12d). However, if the relative transverse extension and displacement are negligibly small ($E \to \infty$, $w_3 \to 0$), then (12d) is not needed and (12c) simplifies to

$$\begin{aligned} & \tilde{p}^3 + \frac{\bar{\lambda}}{2\lambda}\bar{s}^{\beta}|_{\beta} + \tfrac{1}{2}\bar{c}^{\beta}|_{\beta} - \frac{{}_0\lambda_0\mu({}_0\lambda^2 + \gamma_1\lambda^2)L}{6(1-\eta)}\bar{w}_3|^{\alpha\beta}_{\alpha\beta} \\ & \quad + \frac{1}{L}\bar{w}_3|_{\alpha\beta}(\varepsilon^{\alpha\eta}\varepsilon^{\beta\gamma}\phi|_{\gamma\eta} - \bar{F}^{\alpha\beta}) = 0. \end{aligned} \tag{23}$$

Equation (22a) reduces to

$$-2\bar{\lambda}L\bar{w}_3|^{\alpha\beta}_{\alpha\beta} + \frac{2}{G}\bar{s}^{\beta}|^{\alpha}_{\beta\alpha} - \frac{(1-\eta)(1+\gamma)}{2{}_0\lambda_0\mu\lambda\gamma}\bar{s}^{\beta}|_{\beta} + \frac{(1-\eta)}{{}_0\lambda_0\mu}\tilde{p}^{\beta}|_{\beta} = 0. \tag{22b}$$

The three equations, (19b), (22b) and (23), describe the gross deformations in terms of the variables $\bar{s}^{\alpha}|_{\alpha}$, $\bar{w}_3$ and ϕ. However, equation (22b) is identically satisfied if the functions $\bar{s}^{\alpha}|_{\alpha}$ and $\bar{w}_3$ are derived from an invariant χ as follows [13]:

$$\bar{s}^{\beta}|_{\beta} = -\frac{4L\lambda\bar{\lambda}_{\underline{0}}\lambda_{\underline{0}}\mu\gamma}{(1-\eta)(1+\gamma)}\chi|^{\alpha\beta}_{\alpha\beta} - \bar{\lambda}LG\tilde{P}|^{\alpha}_{\alpha}, \tag{24a}$$

$$\bar{w}_3 = \chi - \frac{4\lambda_{\underline{0}}\lambda_{\underline{0}}\mu\gamma}{(1-\eta)G(1+\gamma)}\chi|^{\alpha}_{\alpha} - \tilde{P}, \tag{24b}$$

in which $\tilde{P}$ is a particular integral of

$$\tilde{P}|^{\alpha}_{\alpha} = -\frac{2\lambda\gamma}{\bar{\lambda}LG(1+\gamma)}\bar{p}^{\alpha}|_{\alpha}. \tag{24c}$$

Substituting (24a, b, c) into (23) we obtain a sixth order equation of the form

$$\chi|^{\alpha\beta\gamma}_{\alpha\beta\gamma} - A\chi|^{\alpha\beta}_{\alpha\beta} + \bar{P} + B(\varepsilon^{\alpha\eta}\varepsilon^{\beta\gamma}\phi|_{\gamma\eta} - \bar{F}^{\alpha\beta}).(\chi|_{\alpha\beta} - C\chi|^{\gamma}_{\gamma\alpha\beta} - \tilde{P}|_{\alpha\beta}) = 0, \tag{25}$$

where A, B, and C are constants and $\bar{P}$ is a loading function (see Notation). Then the nonlinear problem of the isotropic plate with inextensible normal is reduced to the determination of the invariants ϕ and χ by the simultaneous solution of (25) and (19b) wherein the right side is expressed in terms of χ by (24b).

BOUNDARY CONDITIONS

The boundary conditions are most readily obtained by examining the virtual work of the edge forces. Let C denote the boundary curve at the edge of the deformed middle surface of the composite plate. Let ${}_{n}\vec{\mathscr{n}}$ and ${}_{n}\vec{\mathscr{m}}$ denote the tension and couple per unit length of the edge and acting at the middle surface of the facing, ${}_{n}\overrightarrow{\delta Y}$ and ${}_{n}\overrightarrow{\delta\theta}$ the virtual displacement and rotation of the middle surface at the edge of the facing. Let $\vec{t}$ denote the shear traction and $\overrightarrow{\delta V}$ the displacement at the edge of the core. Then the virtual work of all edge forces is

$$\delta\Omega = \int_C \left\{ {}_{\underline{0}}\vec{\mathscr{n}}\,.\,{}_{\underline{0}}\overrightarrow{\delta Y} + {}_{\underline{1}}\vec{\mathscr{n}}\,.\,{}_{\underline{1}}\overrightarrow{\delta Y} + {}_{\underline{0}}\vec{\mathscr{m}}\,.\,{}_{\underline{0}}\overrightarrow{\delta\theta} + {}_{\underline{1}}\vec{\mathscr{m}}\,.\,{}_{\underline{1}}\overrightarrow{\delta\theta} + \int_{-1}^{+1} \lambda L\vec{t}\,.\,\overrightarrow{\delta V}\delta\theta^3 \right\} \mathrm{d}C. \tag{26}$$

The rotation of a facing is (see [10])

$$_{n}\vec{\phi} = -\frac{\varepsilon^{\alpha\gamma}}{L}(\bar{w}_3|_{\alpha} \pm w_3|_{\alpha})\vec{a}_{\gamma}.$$

We define gross and relative rotations as follows:

$$\vec{\bar{\phi}} = \tfrac{1}{2}({}_{\underline{0}}\vec{\phi} + {}_{\underline{1}}\vec{\phi}) = -\frac{\varepsilon^{\alpha\gamma}}{L}\bar{w}_3|_{\alpha}\vec{a}_{\gamma}, \tag{27a}$$

$$\vec{\phi} = \tfrac{1}{2}({}_{\underline{0}}\vec{\phi} - {}_{\underline{1}}\vec{\phi}) = -\frac{\varepsilon^{\alpha\gamma}}{L}w_3|_{\alpha}\vec{a}_{\gamma}. \tag{27b}$$

Since we are concerned only with small strains, the base vectors ${}_{\underline{n}}\vec{A}_i$ of the deformed facings differ from the base vectors $\vec{a}_i$ by a rotation; that is

$$ {}_{\underline{n}}\vec{A}_i = \vec{\bar{A}}_i \pm \vec{\phi} \times \vec{\bar{A}}_i \tag{28a} $$

where

$$ \vec{\bar{A}}_i = \vec{a}_i + \vec{\bar{\phi}} \times \vec{a}_i . \tag{28b} $$

Strictly speaking $\vec{A}_3 \neq \vec{\bar{A}}_3$; they differ by the amount of the transverse shear deformation. However, we intend to take account of gross and relative rotations only and to neglect the shear strains as small compared with these rotations. Accordingly, we take

$$ \vec{A}_3 = \vec{\bar{A}}_3 = \vec{a}_3 - \frac{1}{L}\bar{w}_3|_\alpha \vec{a}^\alpha . \tag{29a} $$

Also, from (27) and (28)

$$ {}_{\underline{n}}\vec{A}_3 = \vec{a}_3 - \frac{1}{L}(\bar{w}_3|_\alpha \pm w_3|_\alpha)\vec{a}^\alpha , \tag{29b} $$

$$ {}_{\underline{n}}\vec{A}_\gamma = \vec{a}_\gamma + \frac{1}{L}(\bar{w}_3|_\gamma \pm w_3|_\gamma)\overrightarrow{a_3}, \tag{29c} $$

wherein products of $\bar{w}_3|_\alpha$ and $w_3|_\alpha$ are neglected.

We denote infinitesimal virtual rotations of the facings by

$$ {}_{\underline{n}}\overrightarrow{\delta\phi} = \overrightarrow{\overline{\delta\phi}} \pm \overrightarrow{\delta\phi}, \tag{30a} $$

where, in accordance with (27),

$$ \overrightarrow{\overline{\delta\phi}} = -\frac{1}{L}\varepsilon^{\alpha\gamma}\overline{\delta w}_3|_\alpha \vec{\bar{A}}_\gamma , \tag{30b} $$

$$ \overrightarrow{\delta\phi} = -\frac{1}{L}\varepsilon^{\alpha\gamma}\delta w_3|_\alpha \vec{\bar{A}}_\gamma . \tag{30c} $$

To account for gross and relative rotations it suffices to take the virtual displacement at the edge of the core in the form

$$ \overrightarrow{\delta V} = \overrightarrow{\overline{\delta w}} + \overrightarrow{\delta w}\theta^3 . \tag{30d} $$

Then the displacements at the interfaces are

$$ {}_{\underline{n}}\overrightarrow{\delta V} = \overrightarrow{\overline{\delta w}} \pm \overrightarrow{\delta w}. \tag{30e} $$

The displacement of a particle at the middle surface of a facing is

$$ \begin{aligned} {}_{\underline{n}}\overrightarrow{\delta Y} &= {}_{\underline{n}}\overrightarrow{\delta V} \pm {}_{\underline{n}}\overrightarrow{\delta\phi} \times \left({}_{\underline{n}}\lambda \frac{L}{2} \vec{A}_3 \right) \\ &= \overrightarrow{\overline{\delta\alpha}} \pm \overrightarrow{\delta\alpha}, \end{aligned} \tag{30f} $$

where

$$\overline{\overline{\delta\alpha}} = \bar{\delta}_i\overline{\overline{A}}{}^i, \qquad \overline{\delta\alpha} = \delta_i\overline{\overline{A}}{}^i \tag{30g, h}$$

and

$$\bar{\delta}_\alpha = \overline{\delta w_\alpha} - \frac{\tilde{\bar{\lambda}}}{4}\overline{\delta w_3}|_\alpha - \frac{({}_0\lambda + {}_1\lambda)}{4}\delta w_3|_\alpha, \tag{30i}$$

$$\delta_\alpha = \delta w_\alpha - \frac{\tilde{\lambda}}{4}\delta w_3|_\alpha - \frac{({}_0\lambda + {}_1\lambda)}{4}\overline{\delta w_3}|_\alpha, \tag{30j}$$

$$\bar{\delta}_3 = \overline{\delta w_3}, \qquad \delta_3 = \delta w_3. \tag{30k, l}$$

Similarly we introduce

$$\vec{\bar{n}} = {}_0\vec{n} + {}_1\vec{n}, \qquad \vec{n} = {}_0\vec{n} - {}_1\vec{n}, \tag{31a, b}$$

$$\vec{\bar{m}} = {}_0\vec{m} + {}_1\vec{m}, \quad \vec{m} = {}_0\vec{m} - {}_1\vec{m} \tag{31c, d}$$

Let $\vec{c} = \bar{c}_\alpha\overline{\overline{A}}{}^\alpha$ and $\vec{u} = \bar{u}_\alpha\overline{\overline{A}}{}^\alpha$ denote the unit tangent and normal to C lying in the surface. The kinematic and dynamic variables are expressed in terms of their physical components in the directions of $\vec{c}$, $\vec{u}$ and $\vec{A}_3$ as follows:

$$\overline{\overline{\delta\alpha}} = \bar{\Delta}_1\vec{u} + \bar{\Delta}_2\vec{c} + \bar{\Delta}_3\vec{A}_3 \tag{32a}$$

$$\overline{\delta\alpha} = \Delta_1\vec{u} + \Delta_2\vec{c} + \Delta_3\vec{A}_3 \tag{32b}$$

$$\vec{\bar{n}} = \bar{X}_1\vec{u} + \bar{X}_2\vec{c} + \bar{X}_3\vec{A}_3 \tag{32c}$$

$$\vec{n} = X_1\vec{u} + X_2\vec{c} + X_3\vec{A}_3 \tag{32d}$$

$$\vec{\bar{m}} = \bar{H}_1\vec{u} + \bar{H}_2\vec{c}, \qquad \vec{m} = H_1\vec{u} + H_2\vec{c} \tag{32e, f}$$

$$\vec{\mathscr{S}} = \frac{1}{L}\bar{u}_\alpha\bar{s}^\alpha\vec{A}_3 = \mathscr{S}\vec{A}_3. \tag{32g}$$

According to (1a) the shear traction $\vec{t}$ is

$$\vec{t} = \frac{1}{2\lambda L^2}\bar{u}_\alpha\bar{s}^\alpha\vec{A}_3.$$

We note too that

$$\bar{\delta}_\alpha = \bar{\Delta}_1\bar{u}_\alpha + \bar{\Delta}_2\bar{c}_\alpha, \qquad \delta_\alpha = \Delta_1\bar{u}_\alpha + \Delta_2\bar{c}_\alpha, \tag{33a, b}$$

$$\bar{u}^\alpha = \varepsilon^{\alpha\beta}\bar{c}_\beta, \qquad \bar{c}^\alpha = \varepsilon^{\beta\alpha}\bar{u}_\beta, \tag{33c, d}$$

$$\frac{\partial}{\partial\theta^\alpha} = L\left(\bar{u}_\alpha\frac{\partial}{\partial U} + \bar{c}_\alpha\frac{\partial}{\partial C}\right), \tag{33e}$$

in which C and U denote arc length along curve C and along the normal, respectively.

When equations (30), (31), (32) and (33) are used in (26), we obtain

$$\delta\Omega = \int_C \left[(\bar{X}_1)\bar{\Delta}_1 + (\bar{X}_2)\bar{\Delta}_2 + \left(\bar{X}_3 + \mathscr{S} - \frac{\partial\bar{H}_1}{\partial C}\right)\bar{\Delta}_3 + (X_1)\Delta_1 + (X_2)\Delta_2 + \left(X_3 - \frac{\partial H_1}{\partial C}\right)\Delta_3 - (\bar{H}_2)\frac{\partial\bar{\Delta}_3}{\partial U} - (H_2)\frac{\partial\Delta_3}{\partial U}\right] dC. \tag{34}$$

If the edge is fixed the kinematic variables $\bar{w}_i$, w_i, $\partial\bar{w}_3/\partial U$ and $\partial w_3/\partial U$ vanish; then the variations $\bar{\Delta}_i$, Δ_i, $\partial\bar{\Delta}_3/\partial U$ and $\partial\Delta_3/\partial U$ are set to zero and $\delta\Omega = 0$. If any of the geometrical constraints is removed a value must be assigned to the associated dynamic variable (in parentheses).

The variables $\bar{X}_i$, X_i, $\bar{H}_\alpha$ and H_α can be expressed in terms of $\bar{n}^{\alpha\beta}$, $\tilde{n}^{\alpha\beta}$, $\bar{q}^\alpha$ and q^α. In keeping with previous approximations we retain products of the tensions and rotations, but neglect other nonlinear terms; then

$$L\bar{X}_1 = \bar{u}_\rho \bar{u}_\alpha \bar{n}^{\alpha\rho}, \qquad L\bar{X}_2 = \bar{c}_\rho \bar{u}_\alpha \bar{n}^{\alpha\rho}, \tag{35a, b}$$

$$L\bar{X}_3 = \bar{u}_\alpha \bar{q}^\alpha + \frac{1}{(1+\gamma)L}\bar{u}_\alpha\left[\frac{2\gamma}{\lambda}\tilde{n}^{\alpha\beta} + (1-\gamma)\bar{n}^{\alpha\beta}\right] w_3|_\beta, \tag{35c}$$

$$LX_1 = \frac{1}{1+\gamma}\bar{u}_\beta \bar{u}_\alpha\left[\frac{2\gamma}{\lambda}\tilde{n}^{\alpha\beta} + (1-\gamma)\bar{n}^{\alpha\beta}\right], \tag{35d}$$

$$LX_2 = \frac{1}{1+\gamma}\bar{c}_\beta \bar{u}_\alpha\left[\frac{2\gamma}{\lambda}\tilde{n}^{\alpha\beta} + (1-\gamma)\bar{n}^{\alpha\beta}\right], \tag{35e}$$

$$LX_3 = \frac{1}{\lambda}\bar{u}_\alpha q^\alpha + \frac{1}{L}\bar{u}_\alpha \bar{n}^{\alpha\rho} w_3|_\rho, \tag{35f}$$

$$\bar{H}_1 = -\bar{c}_\beta \bar{u}_\alpha \bar{m}^{\alpha\beta}, \qquad \bar{H}_2 = \bar{u}_\alpha \bar{u}_\beta \bar{m}^{\alpha\beta}, \tag{35g, h}$$

$$\lambda H_1 = -\bar{c}_\beta \bar{u}_\alpha m^{\alpha\beta}, \qquad \lambda H_2 = \bar{u}_\alpha \bar{u}_\beta m^{\alpha\beta}, \tag{35i, j}$$

$$L\mathscr{S} = \bar{u}_\alpha \bar{s}^\alpha. \tag{35k}$$

The foregoing edge resultants can be transformed to components in the directions of the undeformed coordinate lines by means of (28b).

BUCKLING

We next consider instabilities associated with a bifurcation of equilibrium states. Accordingly, we suppose that a prebuckled state exists and that it is associated with small deformations. Variables associated with the prebuckled state will be marked by an asterisk (*). We seek an equilibrium configuration which is itself a small perturbation of the prebuckled configuration. Then the products of rotations in (2a), (3b), (11), (14d, e) and (21a, b) are not needed. However, because of the dominant role of the tensions, products of the tensions and curvatures are needed in the equilibrium equations (12c) and (12d).

We assume that the plates buckle with little extension of their middle surface; then during buckling the increments in $\bar{n}^{\alpha\beta}$ and $\tilde{n}^{\alpha\beta}$ are small compared to their prebuckled values. The displacements become $\overset{*}{\bar{w}}_3 + \bar{w}_3$ and $\overset{*}{w}_3 + w_3$ (no additional markings are needed on the increments). Then, from (12b), (12c) and (12d), the following equilibrium conditions are obtained for the buckled configuration:

$$\tilde{n}^{\alpha\beta}|_\alpha - \frac{1+\gamma}{2\gamma}\bar{s}^\beta = 0, \tag{36a}$$

$$\frac{\bar{\lambda}}{2\lambda}\bar{s}^\beta|_\beta + \bar{m}^{\alpha\beta}|_{\alpha\beta} + \frac{1}{L}\overset{*}{\bar{n}}{}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{1-\gamma}{L(1+\gamma)}\overset{*}{\bar{n}}{}^{\alpha\beta} w_3|_{\alpha\beta} + \frac{2\gamma}{\lambda L(1+\gamma)}\overset{*}{\tilde{n}}{}^{\alpha\beta} w_3|_{\alpha\beta} = 0 \tag{36b}$$

$$\frac{\tilde{\lambda}}{4}\bar{s}^{\beta}|_{\beta} - L^2\sigma^{33} + m^{\alpha\beta}|_{\alpha\beta} + \frac{2\gamma}{L(1+\gamma)}\overset{*}{\bar{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{\lambda(1-\gamma)}{L(1+\gamma)}\overset{*}{\tilde{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{\lambda}{L}\overset{*}{\bar{n}}^{\alpha\beta}w_3|_{\alpha\beta} = 0. \qquad (36c)$$

Equations (2a), (14a), (14b) and (21a or b) (without the products of rotations) serve to express (36a, b, c) in terms of $\bar{s}^{\beta}$, $\bar{w}_3$ and w_3. ($\overset{*}{\bar{n}}^{\alpha\beta}$ and $\overset{*}{\tilde{n}}^{\alpha\beta}$ are supposedly known. They are determined by the solution of the linear equations for the prebuckled configuration.) The equations (36) become

$$\bar{s}^{\beta} - \frac{2\lambda_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu\gamma}{1+\gamma}\bar{B}^{\alpha\beta\gamma\eta}\left[-\tilde{\lambda}L\bar{w}_3|_{\gamma\eta\alpha} + C_{\gamma\mu}\bar{s}^{\mu}|_{\eta\alpha} - \frac{\lambda^2}{3E}\bar{s}^{\mu}|_{\mu\gamma\eta\alpha} - \frac{\tilde{\lambda}L}{2}w_3|_{\gamma\eta\alpha}\right] = 0, \qquad (37a)$$

$$\begin{aligned}&\frac{\tilde{\lambda}}{2\lambda}\bar{s}^{\beta}|_{\beta} - \frac{{}_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu L}{12}\bar{B}^{\alpha\beta\rho\lambda}[({}_{\mathrm{Q}}\lambda^2 + \gamma_{\mathrm{l}}\lambda^2)\bar{w}_3|_{\rho\lambda\alpha\beta} + ({}_{\mathrm{Q}}\lambda^2 - \gamma_{\mathrm{l}}\lambda^2)w_3|_{\rho\lambda\alpha\beta}] + \\ &\quad + \frac{1}{L}\overset{*}{\bar{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{1-\gamma}{L(1+\gamma)}\overset{*}{\bar{n}}^{\alpha\beta}w_3|_{\alpha\beta} + \frac{2\gamma}{\lambda L(1+\gamma)}\overset{*}{\tilde{n}}^{\alpha\beta}w_3|_{\alpha\beta} = 0,\end{aligned} \qquad (37b)$$

$$\begin{aligned}&\frac{\tilde{\lambda}}{4}\bar{s}^{\beta}|_{\beta} - 2ELw_3 - \frac{\lambda_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu L}{12}\bar{B}^{\alpha\beta\rho\lambda}[({}_{\mathrm{Q}}\lambda^2 + \gamma_{\mathrm{l}}\lambda^2)w_3|_{\rho\lambda\alpha\beta} + ({}_{\mathrm{Q}}\lambda^2 - \gamma_{\mathrm{l}}\lambda^2)\bar{w}_3|_{\rho\lambda\alpha\beta}] + \\ &\quad + \frac{2\gamma}{L(1+\gamma)}\overset{*}{\bar{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{\lambda(1-\gamma)}{L(1+\gamma)}\overset{*}{\tilde{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} + \frac{\lambda}{L}\overset{*}{\bar{n}}^{\alpha\beta}\bar{w}_3|_{\alpha\beta} = 0.\end{aligned} \qquad (37c)$$

The critical loads are determined by the characteristic values of $\overset{*}{\bar{n}}^{\alpha\beta}$ and $\overset{*}{\tilde{n}}^{\alpha\beta}$ in the homogeneous equations (37).

If the shell is isotropic, (37a) can be differentiated covariantly and summed as in equation (22). Then equations (37) yield a system of three linear homogeneous partial differential equations in three scalars, $\bar{s}^{\beta}|_{\beta}$, $\bar{w}_3$ and w_3.

INFINITESIMAL THEORY

When the linear versions of (2a), (14a, b) and (21a) are substituted into the linear versions of (12b, c, d) the result is a system of four linear equations which describe a slight flexure in terms of the variables $\bar{s}^{\alpha}$, $\bar{w}_3$ and w_3. If the plate is isotropic, equation (22a) supplants equations (12b). The linear versions of (22a), (12c) and (12d) (after the introduction of (2a) and (14a, b)) constitute a system of three equations which describe a slight flexure of an isotropic plate in terms of three invariants, $\bar{s}^{\alpha}|_{\alpha}$, $\bar{w}_3$ and w_3. The linear versions of (19a) and (19b) govern the small gross extensions of the aelotropic and isotropic plates, respectively, in terms of the Airy stress function ϕ. The linearization uncouples the flexural and extensional problems.

We turn to the isotropic case in which $\lambda^2 G/E \ll 1$ and $w_3 \ll \bar{w}_3$. Then it is reasonable to approximate (21b) as follows:

$$\tilde{n}^{\alpha\beta} = \lambda_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu\left[-2\tilde{\lambda}L\left(\bar{w}_3|^{\alpha\beta} + \frac{\eta}{1-\eta}a^{\alpha\beta}\bar{w}_3|^{\eta}_{\eta}\right) + \frac{1}{G}\left(\bar{s}^{\beta}|^{\alpha} + \bar{s}^{\alpha}|^{\beta} + \frac{2\eta}{1-\eta}a^{\alpha\beta}\bar{s}^{\eta}|_{\eta}\right)\right]. \qquad (38)$$

If (38) is substituted into (12b) we have

$$\bar{s}^{\beta} - \frac{2\lambda_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu\gamma}{G(1+\gamma)}\left(\bar{s}^{\beta}|^{\alpha}_{\alpha} + \frac{1+\eta}{1-\eta}\bar{s}^{\alpha}|^{\beta}_{\alpha}\right) = \frac{2\lambda_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu\gamma}{1+\gamma}\left(\frac{\bar{p}^{\beta}}{{}_{\mathrm{Q}}\lambda_{\mathrm{Q}}\mu} - \frac{2\tilde{\lambda}L}{1-\eta}\bar{w}_3|^{\alpha\beta}_{\alpha}\right). \qquad (39)$$

If (14a) is substituted into (12c) with the appropriate form of $\bar{B}^{\alpha\beta\gamma\eta}$, we obtain

$$\frac{{}_0\lambda_0\mu L({}_0\lambda^2+\gamma_1\lambda^2)}{6(1-\eta)}\bar{w}_3|^{\alpha\beta}_{\alpha\beta}-\bar{p}^3-\frac{\bar{\lambda}}{2\lambda}\hat{s}^{\beta}|_{\beta}-\tfrac{1}{2}\bar{c}^{\beta}|_{\beta}=0. \tag{40}$$

If (39) is differentiated covariantly and summed and $\hat{s}^{\beta}|_{\beta}$ is eliminated by means of (40), we obtain

$$\bar{w}_3|^{\alpha\beta\gamma}_{\alpha\beta\gamma}-A\bar{w}_3|^{\alpha\beta}_{\alpha\beta}+B\bar{\bar{P}}=0, \tag{41}$$

wherein

$$\bar{\bar{P}}=L\left[\bar{p}^3+\tfrac{1}{2}\bar{c}^{\beta}|_{\beta}-\frac{4\lambda_0\lambda_0\mu\gamma}{(1-\eta)G(1+\gamma)}(\bar{p}^3|^{\alpha}_{\alpha}+\tfrac{1}{2}\bar{c}^{\beta}|^{\alpha}_{\alpha\beta})+\bar{\lambda}\left(\frac{\gamma}{1+\gamma}\right)\bar{p}^{\beta}|_{\beta}\right].$$

Equation (41) together with (40), (38), (14a) and (2b) govern the small gross bending of the isotropic plate.

If the facings are thin enough i.e. ${}_{\eta}\lambda \ll \lambda$, equation (40) may be approximated by

$$\hat{s}^{\beta}|_{\beta}=-\bar{p}^3-\tfrac{1}{2}\bar{c}^{\beta}|_{\beta}.$$

When this is substituted into (38) and (39) we have

$$\begin{aligned}\bar{n}^{\alpha\beta}=&-4\lambda^2{}_0\lambda_0\mu L\left(\bar{w}_3|^{\alpha\beta}+\frac{\eta}{1-\eta}a^{\alpha\beta}\bar{w}_3|^{\eta}_{\eta}\right)\\&+\frac{\lambda_0\lambda_0\mu}{G}(\hat{s}^{\beta}|^{\alpha}+\hat{s}^{\alpha}|^{\beta})-\frac{2\eta\lambda_0\lambda_0\mu}{(1-\eta)G}a^{\alpha\beta}(\bar{p}^3+\tfrac{1}{2}\bar{c}^{\beta}|_{\beta}),\end{aligned} \tag{42}$$

$$\hat{s}^{\beta}-\frac{2\lambda_0\lambda_0\mu\gamma}{G(1+\gamma)}\hat{s}^{\beta}|^{\alpha}_{\alpha}=\frac{2\lambda_0\lambda_0\mu\gamma}{1+\gamma}\left[\frac{\bar{p}^{\beta}}{{}_0\lambda_0\mu}-\frac{1+\eta}{(1-\eta)G}(\bar{p}^3|^{\beta}+\tfrac{1}{2}\bar{c}^{\alpha}|^{\beta}_{\alpha})-\frac{4\lambda L}{1-\eta}\bar{w}_3|^{\alpha\beta}_{\alpha}\right], \tag{43}$$

and upon differentiating and summing,

$$\bar{w}_3|^{\alpha\beta}_{\alpha\beta}=\frac{(1-\eta)(1+\gamma)}{8\lambda^2{}_0\lambda_0\mu L^2\gamma}\bar{\bar{P}}. \tag{44}$$

Equations (42), (43) and (44) are similar to those for the theory of one-layer plates presented by Reissner [12]; they are comparable to equations (7.7.13), (7.7.14) and (7.7.15) of Reference 11.

If the derivatives of the loads are negligible, equations (43) and (44) reduce to

$$\hat{s}^{\beta}-\frac{2\lambda_0\lambda_0\mu\gamma}{G(1+\gamma)}\hat{s}^{\beta}|^{\alpha}_{\alpha}=\frac{2\lambda_0\lambda_0\mu\gamma}{1+\gamma}\left(\frac{\bar{p}^{\beta}}{{}_0\lambda_0\mu}-\frac{4\lambda L}{1-\eta}\bar{w}_3|^{\alpha\beta}_{\alpha}\right), \tag{45}$$

and

$$\bar{w}_3|^{\alpha\beta}_{\alpha\beta}=\frac{(1-\eta)(1+\gamma)}{8\lambda^2{}_0\lambda_0\mu L\gamma}\bar{p}^3. \tag{46}$$

The integration of equations (45) and (46) is discussed in [11].

ILLUSTRATIVE PROBLEM OF BUCKLING

Perhaps the most important application of the foregoing theory is its use in the calculation of buckling loads. To illustrate this we consider two fundamental problems, the cylindrical buckling of a wide plate under (1) unaxial compression and (2) pure bending.

A cross-section of the plate is shown in Fig. 1; the dimensionless coordinates are $\theta_\alpha = x_\alpha/L$ in which x_α denotes a length as shown. For an isotropic plate the elastic constants are

$$\eta\bar{B}^{1111} = \eta\bar{B}^{2222} = \bar{B}^{1122} = \bar{B}^{2211} = \frac{2\eta}{1-\eta}, \qquad \bar{B}^{1212} = 1.$$

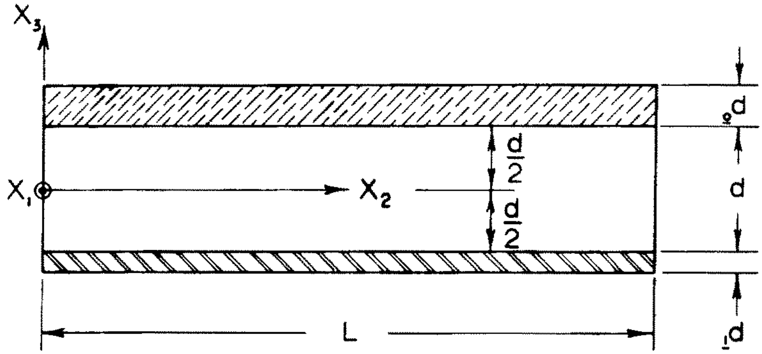

FIG. 1. Cross-section of a sandwich plate.

The plate is supposed to be wide in the direction of x_1 so that the plane strain assumption is justified, i.e. $\gamma_{i1} = 0$. Then

$$\check{s}^1 = \bar{n}^{12} = n^{12} = \bar{m}^{12} = m^{12} = 0$$

and the remaining variables are independent of θ_1.

(1) *Uniaxial compression*

Prior to buckling the only non-zero stress resultant is $\overset{*}{\bar{n}}{}^{22} = -X$. Then equations (37a, b, c) give the following three homogeneous differential equations:

$$\bar{\lambda}L\bar{w}_{3,222} + \frac{L}{2}({}_{\underline{0}}\lambda - {}_{\underline{1}}\lambda)w_{3,222} - C_{22}\check{s}^2_{,22} + \frac{\lambda^2}{3E}\bar{s}^2_{,2222} + \frac{(1-\eta^2)(1+\gamma)}{4\lambda_{\underline{0}}\lambda_{\underline{0}}\mu\gamma}\check{s}^2 = 0, \tag{47a}$$

$$6\frac{\bar{\lambda}}{\lambda}\check{s}^2_{,2} - 12\frac{X}{L}\bar{w}_{3,22} + 12\frac{(\gamma-1)X}{L(\gamma+1)}w_{3,22}$$
$$-\frac{2L_{\underline{0}}\lambda_{\underline{0}}\mu}{1-\eta}({}_{\underline{0}}\lambda^2 + {}_{\underline{1}}\lambda^2\gamma)\bar{w}_{3,2222} - \frac{2L_{\underline{0}}\lambda_{\underline{0}}\mu}{1-\eta}({}_{\underline{0}}\lambda^2 - {}_{\underline{1}}\lambda^2\gamma)w_{3,2222} = 0, \tag{47b}$$

$$3({}_{\underline{0}}\lambda - {}_{\underline{1}}\lambda)\check{s}^2_{,2} - 24ELw_3 + \frac{12\lambda(\gamma-1)}{L(\gamma+1)}X\bar{w}_{3,22} - \frac{12\lambda}{L}Xw_{3,22}$$
$$-\frac{2\lambda_{\underline{0}}\lambda_{\underline{0}}\mu L}{1-\eta}({}_{\underline{0}}\lambda^2 + {}_{\underline{1}}\lambda^2\gamma)w_{3,2222} - \frac{2\lambda_{\underline{0}}\lambda_{\underline{0}}\mu L}{1-\eta}({}_{\underline{0}}\lambda^2 - {}_{\underline{1}}\lambda^2\gamma)\bar{w}_{3,2222} = 0. \tag{47c}$$

For the simply supported edges as depicted in Fig. 2, the boundary conditions are (see equation (35))

$$\left.\begin{aligned} \bar{w}_3 &= w_3 = 0 \\ \bar{m}^{22} &= m^{22} = 0 \\ \bar{n}^{22} &= 0 \end{aligned}\right\} \quad \text{at} \quad \theta_2 = 0, 1. \qquad \begin{aligned} &(48\text{a, b}) \\ &(48\text{c, d}) \\ &(48\text{e}) \end{aligned}$$

Condition (48e) implies that the load acts through the effective centroid of the cross-section as shown in Fig. 2.

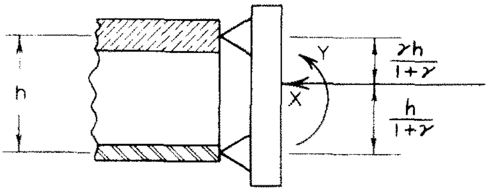

FIG. 2. Edge support.

From the stress–strain relations (14a, b) and the end conditions (48c, d) we have

$$\bar{w}_{3,22} = w_{3,22} = 0 \qquad \text{at} \qquad \theta_2 = 0, 1, \tag{49a, b}$$

and from (21b) and (48e)

$$\frac{\lambda^2}{3E}\bar{s}^2_{,222} - \frac{1}{G}\bar{s}^2_{,2} = 0 \qquad \text{at} \qquad \theta_2 = 0, 1. \tag{50}$$

The boundary conditions are satisfied by the functions

$$\bar{w}_3 = A \sin \alpha_m \theta_2, \tag{51a}$$

$$w_3 = B \sin \alpha_m \theta_2, \tag{51b}$$

$$\bar{s}^2 = C \cos \alpha_m \theta_2, \tag{51c}$$

where $\alpha_m = m\pi$. The substitution of (51a, b, c) into (47a, b, c) leads to three homogeneous algebraic equations in A, B, C, which, in turn, give the characteristic equation,

$$\begin{aligned} p_{\pm} = \frac{(\alpha_m\bar{\lambda})^2}{8K_m}\Bigg[& 1 + \frac{{}_0a^2}{3}(1+b^2)K_m + \left(\frac{1+\gamma}{\gamma}\right)\frac{qK_m}{(\alpha_m\bar{\lambda})^4} + \left(\frac{\gamma-1}{\gamma+1}\right)\tilde{a} + \frac{\tilde{a}^2}{4} \pm \sqrt{\Bigg\{\left[1 + \left(\frac{1+\gamma}{\gamma}\right)\frac{qK_m}{(\alpha_m\bar{\lambda})^4}\right]^2} \\ & - \frac{16qK_m}{(1+\gamma)(\alpha_m\bar{\lambda})^4} + \frac{2(\gamma-1)}{3\gamma}\frac{\tilde{a}({}_0a + {}_1a)qK_m^2}{(\alpha_m\bar{\lambda})^4} + \frac{\tilde{a}^2({}_0a+{}_1a)^2K_m^2}{9} + \frac{2(\gamma-1)}{3(\gamma+1)}\tilde{a}({}_0a+{}_1a)K_m \\ & + 2\left(\frac{\gamma+1}{\gamma}\right)\left[\left(\frac{\gamma-1}{\gamma+1}\right) + \frac{\tilde{a}}{4}\right]\frac{\tilde{a}qK_m}{(\alpha_m\bar{\lambda})^4} + 2\left[\left(\frac{\gamma-1}{\gamma+1}\right) + \frac{\tilde{a}}{4}\right]\tilde{a} + \left[\left(\frac{\gamma-1}{\gamma+1}\right) + \frac{\tilde{a}}{4}\right]^2\tilde{a}^2 \\ & + \left[1 + \left(\frac{\gamma-1}{\gamma+1}\right)\frac{\tilde{a}}{4}\right]\tfrac{2}{3}\tilde{a}^2({}_0a+{}_1a)K_m\Bigg\}\Bigg], \end{aligned} \tag{52}$$

wherein

$$p = -\frac{(1-\eta^2)\,{}_0\sigma^{22}}{{}_0E}, \tag{53a}$$

$$q = \frac{2(1-\eta^2)E}{a_{\underline{0}}a_{\underline{0}}E}, \tag{53b}$$

$$K_m = 1+\frac{4}{q}\left[\frac{E}{G}+\frac{a^2}{3}(\alpha_m\bar{\lambda})^2\right]\frac{\gamma}{1+\gamma}(\alpha_m\bar{\lambda})^2, \tag{53c}$$

$$a = \frac{\lambda}{\bar{\lambda}}, \qquad {}_{\underline{n}}a = \frac{{}_{\underline{n}}\lambda}{\bar{\lambda}}, \qquad \tilde{a} = \frac{\tilde{\lambda}}{\bar{\lambda}}, \qquad b = \frac{{}_{\underline{1}}\lambda}{{}_{\underline{0}}\lambda}, \tag{53d–g}$$

and ${}_{\underline{0}}\sigma^{22}$ is the critical stress in the upper facing. The formula (52) gives two values, p_+ and p_-, according to the sign preceding the radical. The former, p_+, corresponds to modes which are nearly symmetric with respect to the middle surface, i.e. $w_3 > \bar{w}_3$ or $\bar{w}_3 = 0$ if $\gamma = 1$; the latter p_-, is largely antisymmetric, i.e. $\bar{w}_3 > w_3$ or $w_3 = 0$ if $\gamma = 1$. When $\gamma = 1$, equation (52) reduces to the result obtained by Tu [15].

For sufficiently small $\alpha_m\bar{\lambda}$ equation (52) gives

$$p_- \doteq \frac{(\alpha_m\bar{\lambda})^2}{1+\gamma}\left[\frac{\gamma}{1+\gamma}+\frac{1}{12}({}_{\underline{0}}a^2+\gamma\,{}_{\underline{1}}a^2)\right], \tag{54}$$

which is the Euler buckling load; it is also exact for all $\alpha_m\bar{\lambda}$ if $E = G = \infty$.

For large $\alpha_m\bar{\lambda}$ the formula gives

$$p_\pm \doteq \frac{(\alpha_m\bar{\lambda})^2}{8}\left[(1\pm 1)\frac{{}_{\underline{0}}a^2}{3}+(1\mp 1)\frac{{}_{\underline{1}}a^2}{3}\right], \tag{55}$$

which is the buckling load for the top (+) or bottom (−) facing acting independently.

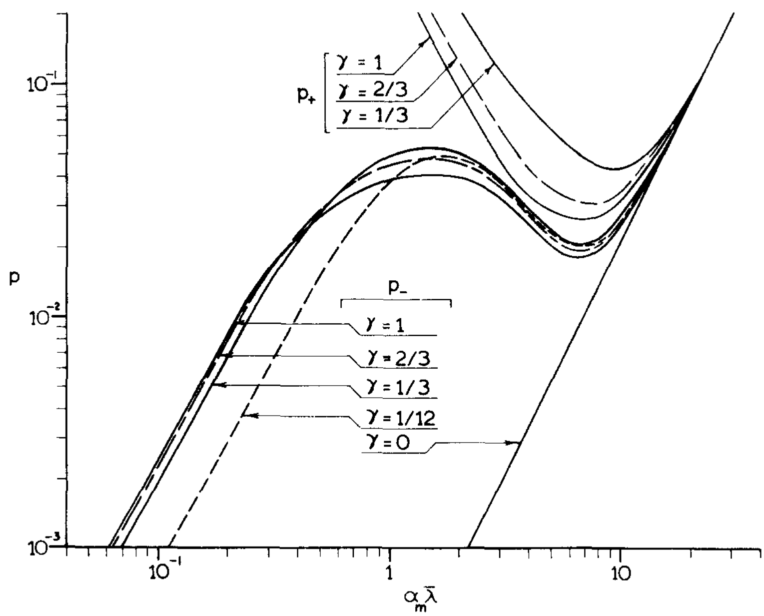

FIG. 3. Buckling under axial load.

Another simplified version of (52) applies when ${}_{\eta}a \ll 1$ and $G \ll E$; then

$$p_{-} \doteq \frac{\pi^2\bar{\lambda}^2\gamma}{(1+\gamma)^2(1+\delta)}, \qquad \text{where} \qquad \delta = \frac{{}_{0}E_{0}a\pi^2\bar{\lambda}^2}{(1-\eta^2)G}\left(\frac{\gamma}{1+\gamma}\right). \tag{56a, b}$$

The factor $(1+\delta)^{-1}$ is the shear correction to the Euler load when the facings are effectively membranes.

The formula (52) can be used confidently for quasi-Euler buckling of thin plates. However, thick plates may buckle in nonsinusoidal modes at lower loads [14]. Consider an example in which

$$\bar{\lambda} = 0, \qquad {}_{0}a = \tfrac{1}{20}, \qquad \frac{{}_{0}E}{E} = 100, \qquad \frac{E}{G} = 2, \qquad \eta = \tfrac{1}{3}.$$

Plots of $p_{\pm}$ versus $\alpha_m\bar{\lambda}$ are shown in Fig. 3. In this instance the curves for the symmetric modes (p_{+}) lie entirely above those of the antisymmetric modes so that the former do not concern us. With $\gamma = 1$ the curve of p_{-} versus $\pi\bar{\lambda}$ is redrawn in Fig. 4 for $m = 1, 2, \ldots$ From that figure it appears that the buckling load is practically constant in the range A to B. However, this implies that a short plate, say $\pi\bar{\lambda} = 6$, buckles as readily as a much longer plate, say $\pi\bar{\lambda} = 0{\cdot}4$. This is unlikely. Instead we anticipate a transition from the Euler buckling to nonsinusoidal antisymmetric modes as $\pi\bar{\lambda}$ increases. According to Goodier and Hsu [14] these modes involve wrinkling near the ends while the central portion remains nearly undeformed. Their work indicates that the buckling load can be significantly lower (about one half) in this range. Then the actual curve will lie below line AB of Fig. 4; this is indicated by the shaded region.

(2) *Pure bending*

If the prebuckled state is pure bending in the direction of θ_2 only, then

$$\overset{*}{\bar{n}}{}^{22} = -\frac{(1+\gamma)Y}{\lambda_{0}\lambda_{0}\mu\gamma(1+s)}, \qquad \overset{*}{\bar{n}}{}^{22} = 0,$$

where Y is the bending couple and

$$s = \frac{1+\gamma}{12\gamma}({}_{0}a^2 + {}_{1}a^2\gamma).$$

If the pure bending state is replaced by the condition illustrated in Fig. 2, then $s = 0$. Accordingly, the results apply as well to the condition of Fig. 2 with the understanding that the applied couple is $Y/(1+s)$ and the critical stresses must be reduced proportionately.

With the above values for $\overset{*}{\bar{n}}{}^{22}$ and $\overset{*}{\bar{n}}{}^{22}$, equations (37a, b, c) yield three homogeneous differential equations much like (47a, b, c). We take the end conditions of (48a, b), (49a, b) and (50); they are illustrated in Fig. 2. As before we have solutions given by (51a, b, c). Substituting them into the differential equations we obtain three linear homogeneous

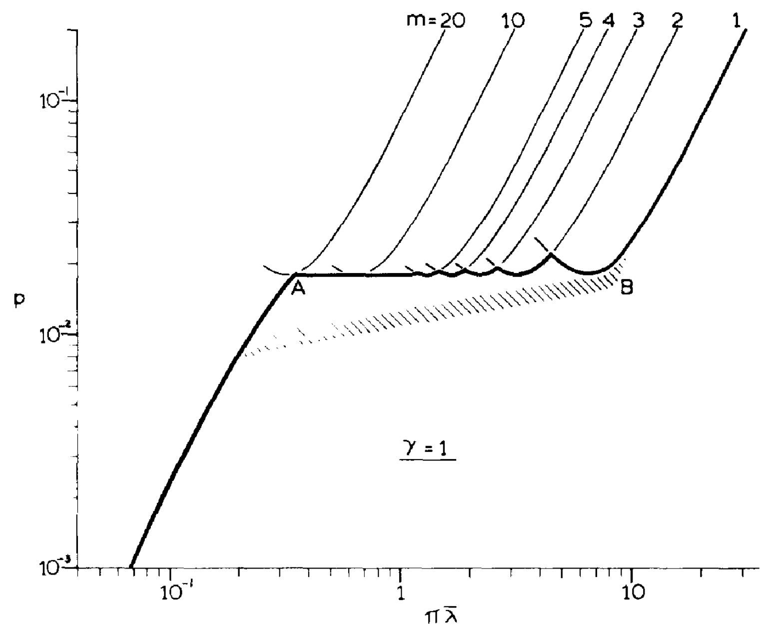

FIG. 4. Axial buckling load vs. thickness parameter.

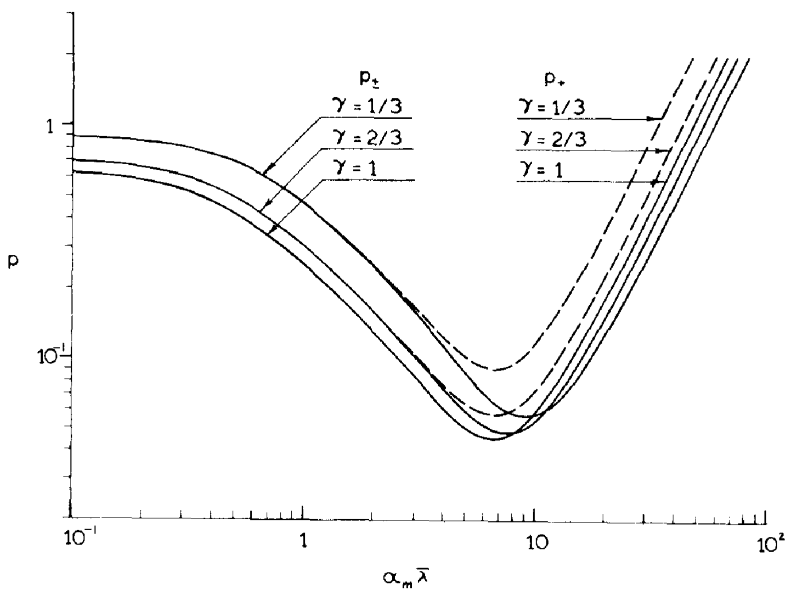

FIG. 5. Buckling under a bending couple.

algebraic equations; the requirement for a nontrivial solution yields a formula for the buckling load,

$$p_\pm = \frac{1}{4}\left(1+\frac{{}_0a}{2}+\frac{{}_1a}{2}\right)(\alpha_m\bar{\lambda})^2\left[\left(\frac{1+\gamma}{6\gamma}\right)({}_0a^2-\gamma\,{}_1a^2) + \frac{\tilde{a}}{K_m} \pm \sqrt{\left\{\left(\frac{1+\gamma}{\gamma}\right)\left[\tfrac{1}{3}({}_0a^2+\gamma\,{}_1a^2)+\frac{4q}{(\alpha_m K_m)^4}\right]\cdot\left[\left(\frac{1+\gamma}{12\gamma}\right)({}_0a^2+\gamma\,{}_1a^2)+\frac{1}{K_m}\right]+\frac{\tilde{a}^2}{K_m^2} + \left(\frac{1+\gamma}{12\gamma}\right)\frac{\tilde{a}^2({}_0a^2+\gamma\,{}_1a^2)}{K_m}\right\}}\,\right] \qquad (58)$$

wherein

$$p = (1-\eta^2)\left(\frac{{}_1\sigma^{22}}{{}_1E}-\frac{{}_0\sigma^{22}}{{}_0E}\right),$$

and ${}_0\sigma^{22}$ and ${}_1\sigma^{22}$ denote the stresses at the uppermost and lowermost points of the plate.

For small values of $\alpha_m\bar{\lambda}$ the formula gives

$$p_\pm \doteq \pm\frac{1}{2}\left(1+\frac{{}_0a}{2}+\frac{{}_1a}{2}\right)\sqrt{\left\{\left(\frac{1+\gamma}{\gamma}\right)\left[\left(\frac{1+\gamma}{12\gamma}\right)({}_0a^2+\gamma\,{}_1a^2)+1\right]q\right\}}.$$

Here the (+) and (−) signs merely indicate opposite directions of the bending couple. Notice that this value does not depend on $\bar{\lambda}$. Moreover, for long wave lengths ($\pi\bar{\lambda} \ll 1$) we anticipate that the critical stress will be unaffected by the end conditions.

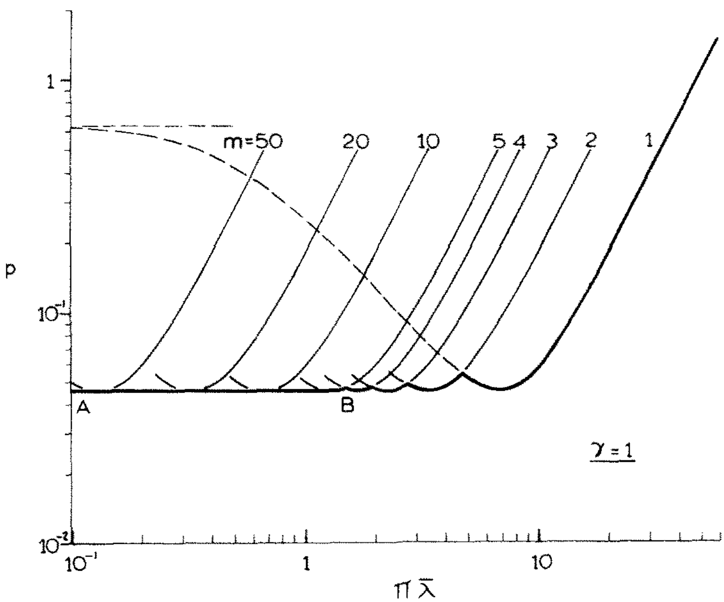

FIG. 6. Buckling couple versus thickness parameter.

For large values of $\alpha_m\bar{\lambda}$ the formula gives

$$p_{\pm} \doteq \frac{1}{24}\left(1+\frac{{}_{\underline{0}}a}{2}+\frac{{}_{\underline{1}}a}{2}\right)(\alpha_m\bar{\lambda})^2\left(\frac{1+\gamma}{\gamma}\right)[(1\pm 1)_{\underline{0}}a^2-(1\mp 1)\gamma_{\underline{1}}a^2].$$

The two values, p_+ and p_-, correspond to the independent buckling of the upper and lower facings under axial compression.

Buckling under conditions of pure bending has fundamental importance as a mechanism of local failure in thin sandwich plates and shells. An estimate of the critical bending stress can be obtained from a plot of p versus $\alpha_m\bar{\lambda}$ as shown in Fig. 5. For purposes of illustration, plots of p versus $\pi\bar{\lambda}$ are shown in Fig. 6 for the case $\gamma = 1$ and $m = 1, 2, \ldots$ From Fig. 6 it is evident that the load on a thin plate cannot exceed that of line AB.

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Zusammenfassung—Die Gleichungen zur Erfassung grosser Durchbiegungen an Schichtplatten mit schwacher Füllung werden abgeleitet und in invarianter Form dargestellt. Die allgemeinen Gleichungen umfassen sowohl die Biegesteifigkeit der Aussenschichten, als auch die Querdehnung des Kernes. Die Gleichungen für Ausbeulen und kleine Durchbiegungen erhält man als Spezialfälle. In einem Beispiel wird die Beullast dilchbestimmt und damit die Anwendung der Gleichungen erläutert.

Абстракт—Выводятся уравнения для больших прогмбов в слоистых плитах со слабым внутренним слоем; уравнения даются в инвариантной форме. Общее уравнение включает сопротивление изгибу внешних поверхностей и поперечное расширение сердцевины. Как специальные случаи даются уравнения для выпучивания и малых прогибов. Иллюстрируется на примере применение этих уравнений для предсказания критических нагрузок.